

A Correctness of Algorithm 2

In this appendix we show:

Theorem A.1. *For any graph G and any pair of vertices s and t in the same connected component of G , Algorithm 2 computes a basis for the s - t path space.*

Proof: We adopt the conventions of Section 2, so that G is a bipartite graph with m blue vertices, n red ones, and e edges oriented from blue to red. Recall the isomorphism, observed in the proof of Theorem 2.6 of the \mathbb{Z} -group of the polynomials $L_C(\cdot)$ and the oriented cycle space $H_1(G, \mathbb{Z})$.

Let c , be the number of connected components of G and define $\beta_1(G) = e - n - m + c$ (the first Betti number of the graph). Some standard facts are that: (i) the rank of $H_1(G, \mathbb{Z})$ is $\beta_1(G)$; (ii) we can obtain a basis for $H_1(G, \mathbb{Z})$ consisting only of simple cycles by picking any spanning forest F of G and then using as basis elements the fundamental cycles C_e of the edges $e \in E \setminus F$. This justifies step 4.

Let (i, j) be an edge of G . Define an i - j path (shortly, just a path) to be a subgraph P such that, for generic rank one A , $L_P(A) = -x_{(i,j)}$; the set of all i - j paths is the i - j path space. By Theorem 2.6, we can write any path as a \mathbb{Z} -linear combination of $x_{(i,j)}$ and oriented cycles. From this, we see that the rank of the path space is $\beta_1(G) + 1$ and the graph theoretic identification of elements in the path space with subgraphs that have even degree at every vertex except i and j . Thus, if (i, j) is an edge of G , step 5 is justified, completing the proof of correctness in this case.

If (i, j) was not an edge, step 1 guarantees that the dummy copy of (i, j) that we added is not in the spanning tree computed in step 3. Thus, the element $P_{(i,j)} = C_{(i,j)} - x_{(i,j)}$ computed in step 6 is a simple path from i to j . The collection of elements generated in step 6 is independent by the same fact in $H_1(G \cup \{(i, j)\}, \mathbb{Z})$ and has rank $\beta_1(G) + 1$. Since none of the elements put a non-zero coefficient on the dummy generator $x_{(i,j)}$, we are done. \square